F. anal. 20.1.2025

- Info / rep. sides / wiki

A. Convolutions

Obs: (f 2π-per., integr.)

$$S_N(f)(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{inx} = \sum_{n=-N}^{N} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y) e^{-iny}\,dy\right) e^{inx}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y) \underbrace{\left(\sum_{n=-N}^{N} e^{in(x-y)}\right)}_{=D_N(x-y),\ \text{Dirichlet kernel}} dy$$

$$\overset{!}{=} (f * D_N)(x)$$

Def. 1: Convolution $*$ for L-per. f, g:

f, g L-per., R.-int. (or $L^1[0,L]$)

$$(f * g)(x) = \frac{1}{L}\int_0^L f(y) \cdot g(x-y)\,dy$$

Rem. 1:

(a) well-def. if f, g R.-int. on $[0,L]$

(b) Recall: $\int_0^L f(x)\,dx \overset{f\ L\text{-per.}}{=} \int_a^{a+L} f(x)\,dx \quad \forall a \in \mathbb{R}$

(c) $f * g$ (local) average of f w.r.t. window f'n g: e.g.

$g = 1 \Rightarrow f * g(x) = \frac{1}{L}\int_0^L f$

$g = \frac{L}{2h}\chi_{[-h,h]} \Rightarrow f * g(x) = \frac{1}{2h}\int_{-h}^{h} f(x+y)\,dy$

Exercise 2: Use $y \mapsto y + x$ and (b)

<u>Lem. 1:</u> $f, g \in C(\mathbb{S})$, then $\|f*g\|_1 \le \|f\|_1 \cdot \|g\|_1$

(*) $$\|f*g\|_1 \le \frac{1}{2\pi} \|f\|_1 \cdot \|g\|_1$$

<u>Pf.:</u> Exercise 2

<u>Rem. 2:</u>

a) By approx. ($C(\mathbb{S}) \ni f_n \xrightarrow{L^1} f \in L^1(\mathbb{S})$),

(*) holds for $f, g \in L^1(\mathbb{S})$.

Here: $\|f\|_1 = \|f\|_{L^1(\mathbb{S})} = \int_0^{2\pi} |f| \, dx$

b) Hence: $f*g$ well-def. for $f, g \in L^1(\mathbb{S})$

(then $f*g \in L^1(\mathbb{S})$)

c) More generally: $f*g \in L^p$ if $f \in L^1, g \in L^p$

and $\|f*g\|_p \le \frac{1}{2\pi} \|f\|_1 \cdot \|g\|_p$, $p \in [1, \infty]$.

Young's convol'n ineq.

Prop. 1: f, g L-per., R. int.

(a) $f * (Ag + Bh) = A(f*g) + B(f*h)$; $A, B \in \mathbb{C}$

(b) $f*g = g*f$

(c) $(f*g)*h = f*(g*h)$

(d) $\widehat{f*g}(n) = \hat{f}(n) \cdot \hat{g}(n)$

(e) $f*g$ is unif. cont.

Rem 3. p. 6

Need to change order of integration - ok if f (e.g.) cont.:

Thm. 1: $f \in C(R_1 \times R_2)$, $R_i \subset \mathbb{R}^{d_i}$ closed rectangles.

Then $F(x_1) = \int_{R_2} f(x_1, x_2)\,dx_2$ cont. in R_1, and

$$\iint_{R_1 \times R_2} f(x)\,dx = \int_{R_1} \left(\int_{R_2} f(x_1, x_2)\,dx_2 \right) dx_1$$

"Pf.:" (see SS App.)

$$\int_{R_1 \times R_2} f = \inf_{P_1, P_2} \mathcal{U}(P_1 \times P_2, f) = \sup_{P_1, P_2} \mathcal{L}(P_1 \times P_2, f)$$

$$\mathcal{U}(P_1 \times P_2, f) = \sum_S \sum_T \left(\sup_{S \times T} f(x_1, x_2) \right) |S| \cdot |T|$$

subrectangles in R_1, R_2

$\sup_{S \times T} \geq \sup_S (\sup_T)$

$$\geq \sum_S \sum_T \sup_{x_1 \in S} \left(\sup_{x_2 \in T} f \right) |S| \cdot |T|$$

$$= \sum_S \sup_S \left[\sum_T (\sup_T f) |T| \right] |S|$$

$$\geq \int_{R_2} f(x_1; x_2)\,dx_2$$

9:00

$$\geq \mathcal{U}\Big(P_1\,,\ \int_{R_2} f(x_1, x_2)\,dx_2\Big)$$

Hence

$$\mathcal{U}(P_1\times P_2, f) \geq \mathcal{U}\Big(P_1, \int_{R_2} f\,dx_2\Big) \overset{\text{similar arg. as for } \mathcal{U}}{\geq} \mathcal{L}\Big(\bar P_1, \int_{R_2} f\,dx_2\Big) \geq \mathcal{L}(\bar P_1 \times \bar P_2, f)$$

Take sup in P_1, P_2 and inf in $\bar P_1, \bar P_2$ $\square$

Pf: of Prop. 7:

(a) By lin. of $\int_0^L dx$

(e) i) Assume $g \in C[0,L]$

$$|(f*g)(x_1) - (f*g)(x_2)| = \frac{1}{L}\Big|\int_0^L f(y)\big(g(x_1-y) - g(x_2-y)\big)\,dy\Big|$$

$$\leq \frac{1}{L}\int |\ -\ | \leq \frac{1}{L}\|f\|_\infty \int_0^L \omega_g(|x_1-x_2|)\,dx$$

$$= \|f\|_\infty\, \omega_g(|x_1-x_2|) \qquad (\Rightarrow f*g \text{ unif. cont.})$$

ii) g R.-int.:

(i) Take $g_n \in C[0,L]$ s.t.

$$\|g_n\|_\infty \leq \|g\|_\infty\,, \quad \|g_n - g\|_1 \underset{n\to\infty}{\longrightarrow} 0.$$

Then

$$|(f*g)(x_1) - (f*g)(x_2)|$$

$$\underset{\substack{(a)\\ +\Delta\text{-ineq.}}}{\overset{\text{chk.}}{\leq}} \underbrace{|(f*g)(x_1) - (f*g_n)(x_1)|}_{f*(g-g_n)(x_1)} + |(f*g_n)(x_1) - (f*g_n)(x_2)| + \underbrace{|(f*g_n)(x_2) - (f*g)(x_2)|}_{f*(g_n-g)(x_2)}$$

i)

$$\leq \|f\|_\infty \underbrace{\|g-g_n\|_1}_{\substack{\longrightarrow 0 \\ n\to\infty}} + \|f\|_\infty \underbrace{\omega_{g_n}(|x_1-x_2|)}_{\substack{\longrightarrow \\ |x_1-x_2|\to 0}} + \|f\|_\infty \cdot \|g-g_n\|_1$$

First n large, then $|x_1-x_2|$ small. ($\Rightarrow f*g$ unif. cont.)

(b), (c), (d) when $f,g \in C([0,L])$:

(b) $$\int_0^L f(y)\,g(x-y)\,dy \overset{\substack{\tilde y = x-y}}{\underset{d\tilde y = -dy}{=}} -\int_x^{x-L} f(x-\tilde y)\,g(\tilde y)\,d\tilde y$$

$$= \int_{x-L}^{x} f(x-\tilde y)\,g(\tilde y)\,d\tilde y \underset{\substack{\uparrow \\ f,g\ L\text{-per.}}}{=} \int_0^L f(x-\tilde y)\,g(\tilde y)\,d\tilde y$$

(c) Change order of int. + change of var.'s (Exercise 2)

(d) $$L\,\widehat{f*g}(n) = \int_0^L (f*g)(x)\,e^{-inx}\,dx$$

$$= \int_0^L \left(\frac{1}{L}\int_0^L f(y)\,g(x-y)\,dy\right) e^{-inx}\,dx$$

$$\overset{\text{Thm. 1}}{=} \int_0^L f(y)\,e^{-iny} \Big(\frac{1}{L}\underbrace{\int_0^L g(x-y)\,e^{-in(x-y)}\,dx}\Big)\,dy$$

$$\overset{\tilde x = x-y}{=} \int_{-y}^{L-y} g(\tilde x)\,e^{-in\tilde x}\,d\tilde x$$

$$\overset{g\text{ per.}}{=} \int_0^L \cancel{g(\tilde x)\,e^{-in\tilde x}\,d\tilde x} = L\cdot\hat g(n)$$

$$= \int_0^L f(y)\,e^{-iny}\,dy \cdot \hat g(n) = L\,\hat f(n)\cdot\hat g(n)$$

(b), (c), (d) when f,g R-int.:

10:03 Approx. f,g by cont. f'ns, much like in e) $\square$

p. 2

Rem. 3:

i) (a) – (d) holds for $f, g \in L^1(\$)$

(by approximation ...)

ii) (e) $\Rightarrow$ $(f * g)$ more regular than f or g

Also holds for $f \in L^1(\$)$, $g \in L^\infty(\$)$

F. anal. 24.1.2025

- Slides / info - ref grp.

A. Good kernels

Want: $S_N(f) = D_N * f \rightarrow f$

Rem 1:

(0) δ-distr.: $\delta_0(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\delta(x)\,dx \overset{def}{=} f(0)$ $\forall$ f cont. at $x=0$
(Mat 4K)

i) $(f*\delta)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)\,\delta(y)\,dy \overset{by\ def}{=} f(x-0) = f(x)$

ii) $\frac{1}{2\pi}\int_{-\pi}^{\pi} 1\cdot\delta = 1$ ("mass 1")

iii) $\forall \delta>0$ $\int_{\delta\le|x|\le\pi} \delta(x)\,dx = \int_{-\pi}^{\pi} \chi_{\delta\le|x|\le\pi}(x)\,\delta(x)\,dx = \chi_{\delta\le|x|\le\pi}(0) = 0$

("concentrated at $x=0$")

Obs 1:

$\delta * f = f$, so want "$D_N \rightarrow \delta$",

or D_N approx.-δ, $\{D_N\}_N$ δ-seq'nce

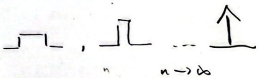
(δ is $*$-identity, and D_N approx. identity)
limit

<u>Def. 1:</u> $\{K_n(x)\}_{n=1}^{\infty}$ <u>good kernels</u> (= δ-seq'ns) if

(a) $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)\,dx = 1 \quad \forall n \geq 1$ ("Mass 1")

(b) $\exists M > 0$ s.t. $\int_{-\pi}^{\pi} |K_n(x)|\,dx \leq M$

(c) $\forall \delta > 0$, $\int_{\delta \leq |x| \leq \pi} |K_n(x)|\,dx \underset{n\to\infty}{\longrightarrow} 0$ ("concentrates at $x=0$")

<u>Ex. 1:</u> $K_n(x) = 2n\,\chi_{[-\frac{1}{n},\frac{1}{n}]}(x)$ 

~~Rem. 1:~~

<u>Rem. 2:</u>

i) If $K_n \geq 0$: (a) $\Rightarrow$ (b)

~~ii) (a) $\Rightarrow$ K_n has "unit mass" (const. fn. mass 2π)~~

~~iii) (c) $\Rightarrow$ mass of K_n concentrates near $x=0$~~

<u>Thm. 1:</u> $\{K_n\}_n$ good kernels, f integrable on $\mathbb{S}$. Then

(a) f cont. at x $\Rightarrow$ $(K_n * f)(x) \underset{n\to\infty}{\longrightarrow} f(x)$

(b) $f \in C(\mathbb{S}) \Rightarrow (f * K_n)(x) \xrightarrow[n\to\infty]{\text{unif.}} f(x)$ on $\mathbb{S}$

<u>Pf.:</u>

$$I_n = |(K_n * f)(x) - f(x)| \underset{\uparrow}{\overset{\text{Def. 1 (a)}}{=}} \frac{1}{2\pi}\Big|\int_{-\pi}^{\pi} K_n(y)\underbrace{\big(f(x-y) - f(x)\big)}_{\leq 2\|f\|_\infty \text{ or } \omega_f(|y|)}dy\Big|$$

$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)K_n(y)\,dy = f(x)$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}|\cdots| \leq \frac{1}{2\pi}\left(\int_{|y|<\delta}|\cdots| + \int_{\delta\leq|y|\leq\pi}|\cdots|\right)$$

$$(*) \quad \leq \frac{1}{2\pi}\Big(\sup_{|y|<\delta}|f(x-y) - f(x)|\underbrace{\int_{|x|\leq\delta}|K_n|}_{\leq M, \text{ by Def. 1 (b)}} + 2\|f\|_\infty \int_{\delta\leq|x|\leq\pi}|K_n|\Big)$$

$= I_{n,1} + I_{n,2}$

(a) Let $\varepsilon > 0$. Since f cont. at x, $\exists\, \delta > 0$ s.t.

$$|y| < \delta \Rightarrow |f(x-y) - f(x)| < \frac{1}{2}\varepsilon \cdot \frac{2\pi}{M}$$

$$\Rightarrow I_{n,1} < \frac{1}{2}\varepsilon$$

Then, by Def 1 (c), take N_δ s.t.

$$n \geq N_\delta \Rightarrow \int_{\delta \leq |y| \leq \pi} |K_n| < \frac{1}{2}\varepsilon \cdot \frac{\pi}{\|f\|_\infty}$$

$$\Rightarrow I_{n,2} < \frac{1}{2}\varepsilon$$

Hence, $\exists\, N$ s.t.

$$n \geq N \Rightarrow I_n < \varepsilon,$$

and $I_n \to 0 \overset{\text{Def. } I_n}{\Longleftrightarrow} K_n * f \to f$ at x

(b) $\|I_n\|_\infty \overset{(*)}{\underset{\substack{f \text{ unif.} \\ \text{cont.}}}{\leq}} \frac{1}{2\pi}\left(\omega_f(\delta) \cdot M + 2\|f\|_\infty \int_{\delta \leq |x| \leq \pi} |K_n|\right)$

take δ s.t. $\omega_f(\delta) < \frac{1}{2}\varepsilon \frac{2\pi}{M}$, n s.t. $\int_{\delta \leq |y| \leq \pi} |K_n| < \frac{1}{2}\varepsilon \frac{\pi}{\|f\|_\infty}$

$\rightsquigarrow \|I_n\|_\infty \underset{n\to\infty}{\longrightarrow} 0 \Longleftrightarrow K_n * f \overset{\text{unif.}}{\longrightarrow} f$. $\square$

Problem: $\{D_N\}_N$ not good kernels:

(1) $\int_{-\pi}^{\pi} |D_N(x)|\, dx \geq c \log N \underset{N\to\infty}{\to} \infty$ [Exercise 3]
(DN oscillates too much!)
$\Rightarrow$ Def. 1 (b) not ok!

But:

(2) $\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(x)\, dx = \frac{1}{2\pi}\sum_{n=-N}^{N} \underbrace{\int_{-\pi}^{\pi} e^{inx}\, dx}_{= \begin{cases} 0, & n \neq 0 \\ 2\pi, & n = 0 \end{cases}} = 1$

(Cancellations from oscillations!)

Solution: (i) Cesàro/Abel limits (ii) more regular f

B. Cesàro means and summation

Conv. of averages:

Ex. 2:

$$s = \sum_{k=0}^{\infty} (-1)^k = 1 - 1 + 1 - 1 + \dots ,$$

$$s_n = \sum_{k=0}^{N} (-1)^k \Rightarrow \{s_1, s_2, s_3, s_4, \dots\}$$
$$= \{0, 1, 0, 1, \dots\},$$

no limit!

But alternates evenly between 0 and 1,

so average $\frac{1}{2}$ could be seen as a "limit".

→ the Cesàro sums of s.

In general:

$$\sum_{k=0}^{\infty} c_k = c_0 + c_1 + c_2 + \dots , \quad c_k \in \mathbb{C}$$

Partial sums: $s_n = \sum_{k=0}^{n} c_k$

nth Cesàro sum/mean: $\sigma_n = \dfrac{s_0 + s_1 + \dots + s_{n-1}}{n}$

mean of partial sums

Def. 2: $\sum c_k$ Cesàro summable to σ if

$$\sigma_n \to \sigma \quad \text{as} \quad n \to \infty.$$

Notation: $C - \sum c_k = \sigma$

Ex. 2: (cont.)

$$\sigma_n = \frac{1+0+1+ \cdots s_{N-1}}{n} = \frac{1}{n}\left(\frac{n}{2}\cdot 1 + \frac{n}{2}\cdot 0 + O(1)\right) \underset{N\to\infty}{\longrightarrow} \frac{1}{2}$$

so $C - \sum_{n=0}^{\infty} (-1)^k = \frac{1}{2}$

Lem. 2:

(a) $\sum_{k=0}^{\infty} c_k = s \underset{(\not\Leftarrow)}{\Longrightarrow} C - \sum_{k=0}^{\infty} c_k = s$

$c_k = \sigma\left(\frac{1}{k}\right)$

(b) $c_k = \sigma\left(\frac{1}{k}\right), k \to \infty$, $C - \sum_{k=0}^{\infty} c_k = \sigma$

$$\Longrightarrow \sum_{k=0}^{\infty} c_k = \sigma$$

Pf.: (a) Let $\varepsilon > 0$.
s_n conv. $\Longrightarrow \exists K$ s.t. $\underbrace{|s_n| \le K}_{\text{bnd.}} \ \forall n$ and $\forall \varepsilon > 0$

(a) Take $\exists N_1$ s.t. $n \geq N_1 \Longrightarrow |s - s_n| < \frac{\varepsilon}{2}$

Then

$$|\sigma_n - s| \le \frac{1}{n}\sum_{k=0}^{n-1} |s_k - s| \underset{n\geq N_1}{\le} \frac{1}{n}\sum_{k=0}^{N_1-1} 2K + \frac{1}{n}\sum_{k=N_1}^{n} \frac{\varepsilon}{2}$$

$$= \frac{N_1}{n} + \underbrace{\frac{n-N_1}{n}}_{\le 1}\cdot\frac{\varepsilon}{2} < \varepsilon$$

for $n > \frac{2N_1}{\varepsilon}$.

~~(b) Exercise 3~~ $\square$

5,5)

Then

$$|\sigma_n - s| = \left| \frac{\sum_{k=0}^{n-1} s_k}{n} - \underset{\frac{1}{n}\sum_{k=0}^{n-1} s}{\underset{\shortparallel}{s}} \right| = \frac{1}{n}\left|\sum_{k=0}^{n-1}(s_k - s)\right|$$

$$\leq \frac{1}{n}\sum_{k=0}^{n-1}|s_n - s| = \frac{1}{n}\sum_{k=0}^{N_1-1}\underbrace{|s_n - s|}_{\leq K+|s|} + \frac{1}{n}\sum_{k=N_1}^{n}\underbrace{|s_n - s|}_{< \frac{\varepsilon}{2}}$$

$$= \frac{1}{n}\cdot N_1(K+|s|) + \underbrace{\frac{n-N_1}{n}}_{\leq 1}\cdot\frac{\varepsilon}{2}$$

$$< \varepsilon$$

if $n > \dfrac{2N_1(K+|s|)}{\varepsilon}$ $\square$

(b) Exercise 3 $\square$